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Local frequency spectra for non-linear wave equations

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Abstract. For linear wave propagation one is often interested more in the local distribution of the wavevectors than in the global spectral distribution (i.e. the Fourier transform). A function that may act as such a local frequency spectrum is the real-valued Wigner distribution function. In this paper this concept of local frequency spectra is generalised to non-linear wave propagation which is governed by a class of non-linear wave equations. This class includes such well known equations as the non-linear Schrödinger equation, the Korteweg-de Vries equation and the Burgers equation. Furthermore the derivation of a transport equation for these local frequency spectra is given on the basis of the dispersion relation for the linearised wave equation. By taking local moments of this transport equation with respect to the frequency variable, an infinite hierarchy of so-called balance equations is constructed. For the non-linear Schrödinger equation the successive conservation laws (in principle, infinitely many) have been calculated straightforwardly from these balance equations.

1. Introduction

As is well known in the theory of linear wave equations it is sometimes convenient to describe a solution $u(\boldsymbol{r}, t)$ not in the space domain, but in the spatial frequency domain by means of its spatial frequency spectrum, i.e. the Fourier transform $\hat{u}(\boldsymbol{p}, t)$ of the function $u(\boldsymbol{r}, t)$, i.e. defined by

$$\hat{u}(\boldsymbol{p}, t) = (2\pi)^{-3/2} \int^{+\infty} u(\boldsymbol{r}, t) \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{r}) \,\mathrm{d}\boldsymbol{r}. \tag{1}$$

This frequency spectrum describes the global spectral density of the wavevectors $\boldsymbol{p}$ of the individual plane waves composing $u(\boldsymbol{r}, t)$ according to the Fourier synthesis. (Of course the temporal frequency description is also used very often. In the present paper, however, we are interested mainly in the spatial frequency description.) It is, however, sometimes more interesting to study linear wave propagation by means of a local spectral distribution of these wavevectors (Bastiaans 1978). A function that may act as such a local frequency spectrum is the real-valued Wigner distribution function $W(\boldsymbol{r}, \boldsymbol{p}, t)$ (Wigner 1932, Mori *et al* 1962) which is defined by

$$W(\boldsymbol{r}, \boldsymbol{p}, t) = \pi^{-3} \int_{-\infty}^{+\infty} \rho(\boldsymbol{r}, \boldsymbol{r}', t) \exp(-2\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{r}') \,\mathrm{d}\boldsymbol{r}' \tag{2}$$

where the function $\rho(\boldsymbol{r}, \boldsymbol{r}', t)$ is given by

$$\rho(\boldsymbol{r}, \boldsymbol{r}', t) = u(\boldsymbol{r}_1, t) u^*(\boldsymbol{r}_2, t) \tag{3}$$

with the averaged coordinate $\boldsymbol{r} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$ and the separation coordinate $\boldsymbol{r}' = \frac{1}{2}(\boldsymbol{r}_1 - \boldsymbol{r}_2)$. The asterisk denotes complex conjugation. The normalisation in (2) has been chosen so as to have

$$\rho(\boldsymbol{r}, \boldsymbol{r}', t) = \int_{-\infty}^{+\infty} W(\boldsymbol{r}, \boldsymbol{p}, t) \exp(+2\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{r}')\, \mathrm{d}\boldsymbol{p} \tag{4}$$

and

$$\hat{u}(\boldsymbol{p}, t)\hat{u}^*(\boldsymbol{p}, t) = |\hat{u}(\boldsymbol{p}, t)|^2 = \int_{-\infty}^{+\infty} W(\boldsymbol{r}, \boldsymbol{p}, t)\, \mathrm{d}\boldsymbol{r}. \tag{5}$$

For $\boldsymbol{r}' = \boldsymbol{0}$ equation (4) becomes

$$|u(\boldsymbol{r}, t)|^2 = \int_{-\infty}^{+\infty} W(\boldsymbol{r}, \boldsymbol{p}, t)\, \mathrm{d}\boldsymbol{p}. \tag{6}$$

Hence $u(\boldsymbol{r}, t)$ can be reconstructed from $W(\boldsymbol{r}, \boldsymbol{p}, t)$, however, up to a constant phase factor. The quantity (6) represents the spatial energy density of the function $u(\boldsymbol{r}, t)$. Further properties of the Wigner distribution function can be found elsewhere (Claasen and Mecklenbräuker 1980a, b).

An additional advantage of the Wigner distribution function is that it can be applied not only to deterministic wave phenomena (Bastiaans 1978) but also to stochastic signals (Bastiaans 1981). In that case the function $\rho(\boldsymbol{r}, \boldsymbol{r}', t)$ in (2) can be interpreted as the two-point correlation function, i.e.

$$\rho(\boldsymbol{r}, \boldsymbol{r}', t) = \langle u(\boldsymbol{r}_1, t) u^*(\boldsymbol{r}_2, t) \rangle \tag{7}$$

where the angle brackets denote the ensemble average. Thus it becomes possible to use the Wigner distribution function to describe, for example, partially coherent light (Bastiaans 1981).

In both cases, the deterministic as well as the stochastic (linear) wave propagation involves a corresponding propagation of the above-described associated local frequency spectrum which is governed by some transport equation (Bastiaans 1979a, b). For waves that propagate in an inhomogeneous (in space and/or time) medium and that satisfy the geometrical-optical restrictions this transport equation reduces to a partial differential equation of first order of Liouville type. The equations for the characteristics of this equation of Liouville type are Hamiltonian in form, with the temporal frequency playing the role of Hamiltonian. Furthermore these characteristics are identical with the ray paths that follow from the eikonal equation (Bremmer 1973, Bastiaans 1979a, b). It should be noted that, if the medium is indeed inhomogeneous in time also, the temporal frequency may no longer be considered as a given parameter. In that case the Wigner distribution function should represent not only the local spatial spectrum (related to the spatial inhomogeneity) but also the momentary temporal spectrum (related to the temporal inhomogeneity).

Until now the concept of local frequency spectra has been applied mainly to linear waves. Bastiaans (1978, 1979a, b, 1980) and Claasen and Mecklenbräuker (1980a, b, c) studied the Wigner distribution function as applied to deterministic, completely coherent, completely incoherent as well as partially coherent optical signals. Furthermore the relevance of the Wigner distribution function as a tool for time-frequency signal analysis is beyond doubt (Claasen and Mecklenbräuker 1980a, b). Also the relation of the Wigner distribution function to other time (place)-frequency signal representations, like Gabor's signal expansion, has been discussed (Claasen and Mecklenbräuker 1980c, Bastiaans 1982). The application of the Wigner distribution function to non-linear wave phenomena has had very little attention until now although it seems relevant. One can think in this connection of the suggestion of Hasegawa and Kodama (1981) to use non-linear effects in monomode fibres to enhance the bit rate. The optical solitons that would propagate in such a fibre are governed by the non-linear Schrödinger equation that is also considered in the present paper. Therefore it is the aim of the present paper to generalise this concept of local frequency spectra to a class of non-linear wave equations that are characterised by the dispersion relations of their associated linearised versions. Furthermore the transport equation for the Wigner distribution function as applied to these non-linear wave equations is constructed and worked out in detail for some well known members of this class, namely the non-linear Schrödinger equation, the Korteweg-de Vries equation, the modified Korteweg-de Vries equation and the Burgers equation. By taking local moments of these transport equations in seven-dimensional $(\boldsymbol{r}, \boldsymbol{p}, t)$ space with respect to the spatial frequency variable, an infinite hierarchy of equations in four-dimensional $(\boldsymbol{r}, t)$ space will be derived. For the non-linear Schrödinger equation it will turn out to be possible to construct from this infinite hierarchy the successive conservation laws. Finally some other promising perspectives of the application of the Wigner distribution function to non-linear wave phenomena are discussed.

2. Transport equations

As has been mentioned in the introduction, linear wave propagation involves a corresponding propagation of the associated distribution function (2). The equation for the latter, a transport equation, must be derived from the underlying model equation. In many cases (and always in quantum mechanics) this equation can be represented by

$$\frac{\partial u}{\partial t} = -F\left(\boldsymbol{r}, \frac{\partial}{\partial \boldsymbol{r}}\right) u \tag{8}$$

where F is some explicit function of the spatial coordinates contained in $\boldsymbol{r}$ and of the partial derivatives contained in the gradient operator. The corresponding transport equation for W can now be obtained by multiplying the equation (8) for $u(\boldsymbol{r}_1, t)$ by $u^*(\boldsymbol{r}_2, t)$ and adding the complex-conjugate expression with $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ interchanged. The result is

$$\frac{\partial}{\partial t}\rho(\boldsymbol{r}, \boldsymbol{r}', t) = -\left[F\left(\boldsymbol{r}_1, \frac{\partial}{\partial \boldsymbol{r}_1}\right) + F^*\left(\boldsymbol{r}_2, \frac{\partial}{\partial \boldsymbol{r}_2}\right)\right]\rho(\boldsymbol{r}, \boldsymbol{r}', t). \tag{9}$$

Both sides of this expression are now multiplied by $\pi^{-3}\exp(-2\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}')$ and the functions F and F^* are expanded in Taylor series around $\boldsymbol{r}$ and $\frac{1}{2}\partial/\partial\boldsymbol{r}$. Finally the resulting equation is integrated over the complete three-dimensional $\boldsymbol{r}'$ space. Thus the transport equation for W is found to be given by

$$\frac{\partial W}{\partial t} = -2\,\mathrm{Re}\left[F\left(\boldsymbol{r} + \frac{\mathrm{i}}{2}\frac{\partial}{\partial \boldsymbol{p}}, \frac{1}{2}\frac{\partial}{\partial \boldsymbol{r}} + \mathrm{i}\boldsymbol{p}\right)\right] W. \tag{10}$$

If the wave phenomenon is governed by some dispersion relation $\omega = \omega(\boldsymbol{r}, \boldsymbol{p})$ in which the $\boldsymbol{r}$ dependence can take account of a possible spatial inhomogeneity of the medium (at least in a WKB approximation), the function F is given by

$$F\left(\boldsymbol{r}, \frac{\partial}{\partial \boldsymbol{r}}\right) = \mathrm{i}\omega\left(\boldsymbol{r}, -\mathrm{i}\frac{\partial}{\partial \boldsymbol{r}}\right). \tag{11}$$

The resulting transport equation (10) for W now becomes

$$\frac{\partial W}{\partial t}=2\,\mathrm{Im}\left[\omega\left(\boldsymbol{r}+\frac{\mathrm{i}}{2}\frac{\partial}{\partial \boldsymbol{p}},\boldsymbol{p}-\frac{\mathrm{i}}{2}\frac{\partial}{\partial \boldsymbol{r}}\right)\right]W \tag{12}$$

or, in symbolic form,

$$\frac{\partial W}{\partial t}=2\sin\left[\frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{p}}\frac{\partial}{\partial \tilde{\boldsymbol{r}}}-\frac{\partial}{\partial \boldsymbol{r}}\frac{\partial}{\partial \tilde{\boldsymbol{p}}}\right)\right](\omega_r(\tilde{\boldsymbol{r}},\tilde{\boldsymbol{p}})\,W(\boldsymbol{r},\boldsymbol{p},t))\Big|_{\substack{\tilde{\boldsymbol{r}}=\boldsymbol{r}\\ \tilde{\boldsymbol{p}}=\boldsymbol{p}}}$$

$$+2\cos\left[\frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{p}}\frac{\partial}{\partial \tilde{\boldsymbol{r}}}-\frac{\partial}{\partial \boldsymbol{r}}\frac{\partial}{\partial \tilde{\boldsymbol{p}}}\right)\right](\omega_i(\tilde{\boldsymbol{r}},\tilde{\boldsymbol{p}})\,W(\boldsymbol{r},\boldsymbol{p},t))\Big|_{\substack{\tilde{\boldsymbol{r}}=\boldsymbol{r}\\ \tilde{\boldsymbol{p}}=\boldsymbol{p}}} \tag{13}$$

where it has been assumed that the dispersion operator can be split into a real and an imaginary part according to

$$\omega(\boldsymbol{r},\boldsymbol{p})=\omega_{\mathrm{r}}(\boldsymbol{r},\boldsymbol{p})+\mathrm{i}\omega_{\mathrm{i}}(\boldsymbol{r},\boldsymbol{p}). \tag{14}$$

The above-described theory for linear wave propagation can be generalised to a special class of non-linear wave phenomena for which the model equation (8) is modified in the sense that the function F now may also depend upon the wavefield itself according to

$$F\left(\boldsymbol{r},\frac{\partial}{\partial \boldsymbol{r}},u(\boldsymbol{r},t)\right)=G\left(\boldsymbol{r},\frac{\partial}{\partial \boldsymbol{r}}\right)+L[u(\boldsymbol{r},t)] \tag{15}$$

where L is some operator acting on $u(\boldsymbol{r},t)$. For a wave equation of which the linearised version can be represented by some dispersion relation $\Omega(\boldsymbol{r},\boldsymbol{p})$ this generalisation means that

$$\omega(\boldsymbol{r},\boldsymbol{p},u(\boldsymbol{r},t))=\Omega(\boldsymbol{r},\boldsymbol{p})-\mathrm{i}L[u(\boldsymbol{r},t)]. \tag{16}$$

The class of equations that is represented by (8) and (15) contains some of the best known non-linear wave equations such as the non-linear Schrödinger equation, the (modified) Korteweg-de Vries equation and the Burgers equation. However, non-evolution equations such as the sine-Gordon equation do not belong to the class of equations that is represented by (8) and (15). As will be seen in § 2.2 this class of equations also contains equations that do not possess soliton solutions.

To illustrate the generalisation we consider some of the well known non-linear wave equations in one spatial dimension.

2.1. The non-linear Schrödinger equation

$$\begin{aligned}
&\frac{\partial u}{\partial t}=-\left[\frac{1}{\mathrm{i}}\left(\frac{\partial^2}{\partial x^2}\pm uu^*\right)\right]u\\
&F=-\mathrm{i}\frac{\partial^2}{\partial x^2}+L[u]\\
&L[u]=\mp\mathrm{i}uu^*=\mp\mathrm{i}|u|^2\\
&\omega=\omega_r=p^2\mp|u|^2.
\end{aligned} \tag{17}$$

The transport equation for W is now constructed by means of (13) and is thus found to be given by

$$\frac{\partial W}{\partial t}+2p\frac{\partial W}{\partial x}\pm 2\sin\left(\frac{\partial^2}{2\partial p\,\partial \tilde{x}}\right)[|u(\tilde{x},t)|^2\,W(x,p,t)]|_{\tilde{x}=x}=0 \tag{18}$$

where

$$|u(x,t)|^2 = \int_{-\infty}^{+\infty} W(x,p,t)\,\mathrm{d}p. \tag{19}$$

It should be noted that in this manner the transport equation for W can be constructed not only for the Schrödinger equation with the above-given cubic non-linearity, but also for other types of non-linearities. For example, the relaxation of the quasi-neutrality assumption in the description of non-linear Langmuir waves and ion-acoustic waves in a plasma and the non-linear coupling between them (Nicholson 1983) is shown to lead to the following expression for the non-linearity L in (17):

$$L[u] = \sum_{n=0}^{N} \lambda_n \frac{\partial^{2n}}{\partial x^{2n}} (|u|^2) \tag{20}$$

where λ_0 is normalised to $+\mathrm{i}$. If all the coefficients λ_n are assumed to be zero (except $\lambda_0 = +\mathrm{i}$) we do have exact neutrality on the slow timescale associated with the slowly varying amplitude of the Langmuir wave. The terms in (20) with the coefficients λ_n $(n = 1, 2, 3, \ldots)$ take account of the deviation from exact neutrality on this slow timescale.

Another generalised non-linear potential one can look at is the following one:

$$L[u] = \sum_{n=1}^{M} \frac{\mu_n}{(2n-1)!} \int_x^{\infty} (x-y)^{2n-1} u(y,t) u^*(y,t)\,\mathrm{d}y. \tag{21}$$

This is identical with

$$\frac{\partial^{2M} L}{\partial x^{2M}} = \sum_{n=1}^{M} \mu_n \frac{\partial^{2M-2n}}{\partial x^{2M-2n}} (|u|^2). \tag{22}$$

For $M = 1$ the transport equation (18) for W together with (22) is a strong reminder of the Vlasov–Poisson system of equations in plasma physics (Nicholson 1983).

It is of course possible to combine (20) and (22) into one generalised non-linear potential that is given by

$$\begin{aligned} \frac{\partial^{2M} L}{\partial x^{2M}} &= \sum_{n=0}^{N} \lambda_n \frac{\partial^{2M+2n}}{\partial x^{2M+2n}} (|u|^2) + \sum_{n=1}^{M} \mu_n \frac{\partial^{2M-2n}}{\partial x^{2M-2n}} (|u|^2) \\ &= \sum_{n=0}^{N+M} \nu_n \frac{\partial^{2n}}{\partial x^{2n}} (|u|^2) \end{aligned} \tag{23}$$

where

$$\begin{aligned} \nu_n &= \lambda_{n-M} && \text{for } n = M, 1+M, \ldots, N+M \\ \nu_n &= \mu_{M-n} && \text{for } n = 0, 1, \ldots, M-1 \text{ and } \nu_M = +\mathrm{i}. \end{aligned} \tag{24}$$

The impact of the above generalised non-linear potential with respect to the instability of the modulation of a uniform wavetrain of which the envelope u is assumed to be governed by the non-linear Schrödinger equation (the Benjamin–Feir instability (Benjamin and Feir 1967)) can be studied on the basis of the transport equation for W in the same manner as Landau (1946) treated the similar-looking Vlasov equation of plasma physics in order to obtain the Landau damping of Langmuir oscillations. Also the long-time behaviour of this modulational instability and its connection with the Fermi–Pasta–Ulam recurrence phenomenon (Janssen 1981) can be investigated by means of the transport equation for W. The details of these investigations will be presented elsewhere.

Finally it is noted that transport equations for W can, in the previously described manner, also be constructed for the whole family of generalisations of the non-linear Schrödinger equation that has already been studied in the literature; see for example Dodd and Fordy (1984). For such a generalised non-linear Schrödinger equation the corresponding transport equation for W does have a Hamiltonian structure. This falls out naturally in terms of the Moyal bracket that is defined by

$$[a, b]_\delta = \sum_{n\geq 0} \frac{\delta^{n-1}}{n!}\left(\frac{\partial^n a}{\partial x^n}\frac{\partial^n b}{\partial p^n} - \frac{\partial^n b}{\partial x^n}\frac{\partial^n a}{\partial p^n}\right) \tag{25}$$

where $a = a(x, p)$ and $b = b(x, p)$. With this Moyal bracket the transport equation for W for the generalised non-linear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} + V(x)u = 0 \tag{26}$$

can be written as follows:

$$\frac{\mathrm{D}W}{\mathrm{D}t} = \frac{\partial W}{\partial t} + [W, \tfrac{1}{2}p^2 - \tfrac{1}{2}V(x)]_{\mathrm{i}/2} - [W, -\tfrac{1}{2}p^2 + \tfrac{1}{2}V(x)]_{\mathrm{i}/2}$$

$$= \frac{\partial W}{\partial t} + [W, p^2 - V(x)]'_1 = 0 \tag{27}$$

where

$$[a, b]'_\delta = \sum_{n\leq 0} \frac{(-1)^n\delta^{2n}}{(2n+1)!}\left(\frac{\partial^{2n+1}a}{\partial x^{2n+1}}\frac{\partial^{2n+1}b}{\partial p^{2n+1}} - \frac{\partial^{2n+1}b}{\partial x^{2n+1}}\frac{\partial^{2n+1}a}{\partial p^{2n+1}}\right). \tag{28}$$

The Moyal bracket then gives the sine term in (18). This point has been discussed by Adler (1979) and Lebedev and Manin (1979). Lebedev and Manin (1979) also discussed the quasi-classical limit ($\delta \to 0$), where the Moyal bracket becomes the canonical Poisson bracket.

2.2. *The generalised Korteweg-de Vries equation*

$$\begin{aligned}
&\frac{\partial u}{\partial t} = -\left(\frac{\partial^3}{\partial x^3} \pm u^q \frac{\partial u}{\partial x}\right)u \qquad q = 0, 1, 2, \ldots \\
&F = \frac{\partial^3}{\partial x^3} + L[u] \\
&L[u] = \pm u^q \frac{\partial u}{\partial x} = \frac{\pm 1}{q+1}\frac{\partial(u^{q+1})}{\partial x} \\
&\omega = \omega_r - \mathrm{i}\omega_i = -p^3 \mp \mathrm{i}u^q \frac{\partial u}{\partial x}.
\end{aligned} \tag{29}$$

For $q = 0$ we thus have the Korteweg-de Vries equation and for $q = 1$ we have the modified Korteweg-de Vries equation. From the dispersion relation we find the following transport equation for W:

$$\frac{\partial W}{\partial t}-3p^2\frac{\partial W}{\partial x}+\frac{1}{4}\frac{\partial^3 W}{\partial x^3}\pm 2\cos\left(\frac{\partial^2}{2\partial p\partial \tilde{x}}\right)\left(u^q(\tilde{x},t)\frac{\partial u(\tilde{x},t)}{\partial \tilde{x}}W(x,p,t)\right)\Bigg|_{\tilde{x}=x}=0 \tag{30}$$

where for real-valued u

$$u^2(x,t)=\int_{-\infty}^{+\infty}W(x,p,t)\,\mathrm{d}p. \tag{31}$$

2.3. The Burgers equation

$$\begin{aligned}
&\frac{\partial u}{\partial t}=-\left(-\frac{\partial^2}{\partial x^2}+\frac{\partial u}{\partial x}\right)u\\
&F=-\frac{\partial^2}{\partial x^2}+L[u]\\
&L[u]=\frac{\partial u}{\partial x}\\
&\omega=\mathrm{i}\omega_{\mathrm{i}}=-\mathrm{i}\left(p^2+\frac{\partial u}{\partial x}\right).
\end{aligned} \tag{32}$$

For the transport equation we now find

$$\frac{\partial W}{\partial t}+2p^2W-\frac{1}{2}\frac{\partial^2 W}{\partial x^2}+2\cos\left(\frac{\partial^2}{2\partial p\,\partial \tilde{x}}\right)\left(\frac{\partial u(\tilde{x},t)}{\partial \tilde{x}}W(x,p,t)\right)\Bigg|_{\tilde{x}=x}=0 \tag{33}$$

where again for real-valued u

$$u^2(x,t)=\int_{-\infty}^{+\infty}W(x,p,t)\,\mathrm{d}p. \tag{34}$$

In the same manner transport equations for local frequency spectra for other kinds of non-linear wave equations also in more than one spatial dimension (e.g. the Kadomtsev–Petviashvili equation (Kadomtsev and Petviashvili 1970) can be constructed.

3. Hierarchy of balance equations

By means of a procedure that is reminiscent of the derivation of an infinite hierarchy of so-called fluid equations from the Vlasov equation in plasma physics, it is possible to obtain a similar infinite hierarchy of equations in four-dimensional $(\boldsymbol{r}, t)$ space from the transport equation for the local frequency spectrum W. This procedure consists of taking local moments of the transport equation with respect to the spatial frequency variable p. For simplicity we shall restrict ourselves to one spatial dimension; the generalisation to three spatial dimensions is straightforward.

Expression (16) is now rewritten as follows:

$$\omega = \omega_r + i\omega_i = \Omega_r(x, p) + L_i[u(x,t)] + i\{\Omega_i(x,p) - L_r[u(x,t)]\} \tag{35}$$

where it has been assumed that the dispersion operator Ω and the non-linearity operator L can be split into a real and an imaginary part according to

$$\Omega = \Omega_r + i\Omega_i \qquad L = L_r + iL_i. \tag{36}$$

Furthermore it is assumed that the medium is spatially homogeneous, i.e. Ω does not depend on x, only on the spatial frequency p. The generalisation to spatially inhomogeneous media is rather laborious but, again, straightforward. With these assumptions expression (13) can be worked out and as a result it is found that

$$\frac{\partial W}{\partial t} = -\sum_{m=0}^{\infty} \frac{(-4)^m}{(2m+1)!} \frac{\partial^{2m+1}\Omega_r}{\partial p^{2m+1}} \frac{\partial^{2m+1} W}{\partial x^{2m+1}} + \sum_{m=0}^{\infty} \frac{(-4)^m}{(2m+1)!} \frac{\partial^{2m+1} L_i}{\partial x^{2m+1}} \frac{\partial^{2m+1} W}{\partial p^{2m+1}}$$
$$+2\sum_{m=0}^{\infty} \frac{(-4)^m}{(2m)!} \frac{\partial^{2m}\Omega_i}{\partial p^{2m}} \frac{\partial^{2m} W}{\partial x^{2m}} - 2\sum_{m=0}^{\infty} \frac{(-4)^m}{(2m)!} \frac{\partial^{2m} L_r}{\partial x^{2m}} \frac{\partial^{2m} W}{\partial p^{2m}}. \tag{37}$$

We also restrict ourselves to the following class of non-linear wave equations:

$$\frac{\partial u}{\partial t} = -\int_{-\infty}^{+\infty} K(x-y)u(y,t)\,dy - L[u(x,t)]u \tag{38}$$

for some kernel K. The dispersion relation of the linearised version of this equation can be found by taking

$$u(x,t) = \exp[i(px - \Omega t)] \tag{39}$$

then we find

$$\Omega(p) = -i\int_{-\infty}^{+\infty} K(x-y)\exp[-ip(x-y)]\,dy$$
$$= -i\int_{-\infty}^{+\infty} K(\xi)\exp(-ip\xi)\,d\xi. \tag{40}$$

So Ω is equal to $-i(2\pi)^{1/2}$ times the Fourier transform of the kernel K. Finally we assume that $K(\xi)$ becomes zero fast enough for $|\xi| \to \infty$ in order to ensure that $\Omega(p)$ is an entire function of p so that Ω can be expanded in a Taylor series that is convergent in the whole complex p plane. So

$$\Omega_r(p) = \sum_{j=0}^{\infty} \alpha_j p^j \qquad \Omega_i(p) = \sum_{j=0}^{\infty} \beta_j p^j \tag{41}$$

where α_j and β_j are real valued and

$$\alpha_j + \mathrm{i}\beta_j = \frac{(-\mathrm{i})^{j+1}}{j!} \int_{-\infty}^{+\infty} \xi^j K(\xi)\, \mathrm{d}\xi. \tag{42}$$

Local moments with respect to the variable p can now be obtained by multiplying equation (37) by p^n with integer n and then integrating from $p = -\infty$ to $p = +\infty$. For that purpose we have to make use of the following formulae:

$$\int_{-\infty}^{+\infty} p^n \frac{\partial^l W}{\partial p^l} \mathrm{d}p = (-1)^l \frac{n!}{(n-l)!} \int_{-\infty}^{+\infty} p^{n-l} W \,\mathrm{d}p \qquad \text{for } l \leqslant n$$

or (43)

$$= 0 \qquad \text{for } l > n$$

provided that

$$\lim_{p\to\pm\infty} p^n \frac{\partial^{n-1} W}{\partial p^{n-1}} = 0 \qquad \text{for } n = 1, 2, \ldots, l. \tag{44}$$

For $l \leqslant n$ we have

$$\int_{-\infty}^{+\infty} p^{n-l} W \,\mathrm{d}p = (2\mathrm{i})^{l-n} \lim_{x'\to 0} \frac{\partial^{n-l}}{\partial x'^{n-l}} [u(x+x', t) u^*(x-x', t)]$$

$$= (2\mathrm{i})^{l-n} D_x^{n-l} u(x, t)\, u^*(x, t) \tag{45}$$

where D_x is Hirota's bilinear operator that is defined on ordered pairs of functions $\sigma(x)$ and $\tau(x)$ as follows:

$$D_x^n \sigma \cdot \tau = \lim_{\varepsilon\to 0} \frac{\partial^n}{\partial \varepsilon^n} [\sigma(x+\varepsilon)\tau(x-\varepsilon)]$$

$$= \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{\partial^{n-l}\sigma}{\partial x^{n-l}} \frac{\partial^l \tau}{\partial x^l}. \tag{46}$$

Thus the moments of (37) are found to be given by

$$\frac{\partial D_n}{\partial t} = -\frac{\partial}{\partial x} \sum_{j=1}^{\infty} (2\mathrm{i})^{1-j} \alpha_j \sum_{m=0}^{E[(j-1)/2]} \binom{j}{2m-1} \frac{\partial^{2m} D_{n+j-2m-1}}{\partial x^{2m}}$$

$$-2\mathrm{i} \sum_{m=0}^{E[(n-1)/2]} \binom{n}{2m+1} \frac{\partial^{2m+1} L_i}{\partial x^{2m+1}} D_{n-2m-1}$$

$$+2 \sum_{j=0}^{\infty} (2\mathrm{i})^{-j} \beta_j \sum_{m=0}^{E(j/2)} \binom{j}{2m} \frac{\partial^{2m} D_{n+j-2m}}{\partial x^{2m}}$$

$$-2 \sum_{m=0}^{E(n/2)} \binom{n}{2m} \frac{\partial^{2m} L_r}{\partial x^{2m}} D_{n-2m} \tag{47}$$

where $n=0, 1, 2, \ldots$, and $E(x)=s$ for $s\leqslant x<s+1$ (s is an integer). D_n is defined by

$$D_n = D_x^n u\cdot u^*. \tag{48}$$

The second term on the right-hand side of (47) has to be interpreted as zero for $n=0$.

If the wave equation is linearised ($L_r = L_i \equiv 0$) and if $\omega = \Omega_r(p)(\Omega_i \equiv 0)$ then the result (47) represents an infinite sequence of conservation laws of the form

$$\frac{\partial T_n}{\partial t}+\frac{\partial X_n}{\partial x}=0 \tag{49}$$

where T_n, the conserved density, and $-X_n$, the flux of T_n, are functionals of u that are given by

$$\begin{aligned} T_n &= D_n \\ X_n &= \sum_{j=1}^{N} (2\mathrm{i})^{1-j}\alpha_j \sum_{m=0}^{E[(j-1)/2]} \binom{j}{2m+1} \frac{\partial^{2m} D_{n+j-2m-1}}{\partial x^{2m}}. \end{aligned} \tag{50}$$

For the non-linear wave equations considered in § 2 we shall work out the balance equations in more detail.

3.1. *The non-linear Schrödinger equation*

$$\begin{aligned} \Omega &= \Omega_r = p^2 \\ L &= \mathrm{i}L_i = \mp\mathrm{i}|u|^2 \end{aligned} \tag{51}$$

so the infinite sequence of balance equations (47) is now given by

$$\begin{aligned} \frac{\partial D_n}{\partial t} - \mathrm{i}\frac{\partial D_{n+1}}{\partial x} \mp 2\mathrm{i}\bigg(n\frac{\partial D_0}{\partial x} D_{n-1} &+ n(n-1)(n-2)\frac{1}{3!}\frac{\partial^3 D_0}{\partial x^3} D_{n-3} \\ &+ n(n-1)(n-2)(n-3)(n-4)\frac{1}{5!}\frac{\partial^5 D_0}{\partial x^5} D_{n-5} \\ &+ \ldots + \left\{ \begin{array}{ll} \left.\dfrac{\partial^n D_0}{\partial x^n} D_0\right) & \text{for } n \text{ odd} \\ \left. n\dfrac{\partial^{n-1} D_0}{\partial x^{n-1}} D_1 \right) & \text{for } n \text{ even} \end{array} \right\} = 0. \end{aligned} \tag{52}$$

As is well known the non-linear Schrödinger equation is completely integrable (Zakharov and Shabat 1972) and possesses an infinite sequence of so-called (Miura *et al* 1968) constants of local conservation type of the form

$$C_n = \int_{-\infty}^{+\infty} T_n(x, t)\,\mathrm{d}x \tag{53}$$

where $T_n(x, t)$ is a polynomial conserved density and

$$\frac{dC_n}{dt} = 0. \tag{54}$$

These constants C_n can be constructed straightforwardly from (52) and the first five of them are thus found to be

$$C_0 = \int_{-\infty}^{+\infty} D_0 \, dx \qquad C_1 = \int_{-\infty}^{+\infty} D_1 \, dx$$

$$C_2 = \int_{-\infty}^{+\infty} (D_2 \pm 2D_0^2) \, dx \qquad C_3 = \int_{-\infty}^{+\infty} (D_3 \pm 6D_0 D_1) \, dx \tag{55}$$

$$C_4 = \int_{-\infty}^{+\infty} (D_4 \pm 8D_0 D_2 \pm 4D_1^2 + 8D_0^3 \mp 4D_{0x}^2) \, dx$$

where subscripts denote partial differentiation. The first three integrals have their usual physical meaning; C_0, C_1 and C_2 are associated, apart from coefficients, with, respectively, the number of particles, the momentum and the energy.

The described technique for finding the invariants C_n is not only applicable to the standard non-linear Schrödinger equation with cubic non-linearity but also, for example, to a Schrödinger equation with a non-linearity of the type given by expression (20) (however, not with a non-linearity of the type given by expression (21)). In order to demonstrate this we have recalculated the first five constants C_n for the following generalised non-linear Schrödinger equation:

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \left(\alpha|u|^2 - \beta \frac{\partial^2 |u|^2}{\partial x^2} \right) u = 0. \tag{56}$$

As a result it is found that

$$C_0 = \int_{-\infty}^{+\infty} D_0 \, dx \qquad C_1 = \int_{-\infty}^{+\infty} D_1 \, dx$$

$$C_2 = \int_{-\infty}^{+\infty} (D_2 + 2\alpha D_0^2 + 2\beta D_{0x}^2) \, dx$$

$$C_3 = \int_{-\infty}^{+\infty} (D_3 + 6\alpha D_0 D_1 - 6\beta D_{0xx} D_1) \, dx \tag{57}$$

$$C_4 = \int_{-\infty}^{+\infty} (D_4 + 8\alpha D_0 D_2 + 4\alpha D_1^2 + 8\alpha^2 D_0^3 - 4\alpha D_{0x}^2 - 36\alpha\beta D_0^2 D_{0xx}$$
$$- 8\beta D_{0xx} D_2 + 4\beta D_{1x}^2 - 4\beta D_{0xx}^2) \, dx.$$

Although with the inclusion of even higher derivatives of even order in (56) the construction of the successive constants C_n is still straightforward, the calculations become rather laborious, which is the reason for the inclusion of only a second-order derivative in the example (56).

3.2. The generalised Korteweg-de Vries equation

$$\Omega = \Omega_r = -p^3 \qquad L = L_r = \frac{\pm 1}{q+1}\frac{\partial(u^{q+1})}{\partial x} \tag{58}$$

where $q = 0, 1, 2, \ldots$. Since now u is assumed to be real valued, D_n is identically zero for n odd. So the infinite sequence of balance equations (47) now becomes

$$\frac{\partial D_{2n}}{\partial t} + \frac{3}{4}\frac{\partial D_{2(n+1)}}{\partial x} + \frac{1}{4}\frac{\partial^3 D_{2n}}{\partial x^3} \pm \frac{2}{q+1}\Bigg(\frac{\partial(u^{q+1})}{\partial x} D_{2n} + 2n(2n-1)\frac{1}{2!}\frac{\partial^3(u^{q+1})}{\partial x^3} D_{2(n-1)}$$
$$+2n(2n-1)(2n-2)(2n-3)\frac{1}{4!}\frac{\partial^5(u^{q+1})}{\partial x^5} D_{2(n-3)}$$
$$+\ldots+\frac{\partial^{2n+1}(u^{q+1})}{\partial x^{2n+1}} D_0\Bigg) = 0. \tag{59}$$

The generalised Korteweg-de Vries equation is known to be completely integrable for $q = 0$ and $q = 1$. A straightforward construction of the successive constants of local conservation type for these values of q from (59) now, however, seems to be impossible or at least as difficult as a derivation directly from the Korteweg-de Vries equation itself.

3.3. The Burgers equation

$$\Omega = \mathrm{i}\Omega_i = -\mathrm{i}p^2 \qquad L = L_r = \frac{\partial u}{\partial x}. \tag{60}$$

Again D_n is identically zero for n odd because u is assumed to be real valued. Hence the balance equations are given by

$$\frac{\partial D_{2n}}{\partial t} - \frac{1}{2} D_{2(n+1)} - \frac{1}{2}\frac{\partial^2 D_{2n}}{\partial x^2} + 2\Bigg(\frac{\partial u}{\partial x} D_{2n} + 2n(2n-1)\frac{1}{2!}\frac{\partial^3 u}{\partial x^3} D_{2(n-1)}$$
$$+2n(2n-1)(2n-2)(2n-3)\frac{1}{4!}\frac{\partial^5 u}{\partial x^5} D_{2(n-2)} + \ldots + \frac{\partial^{2n+1} u}{\partial x^{2n+1}} D_0\Bigg) = 0. \tag{61}$$

There are now no constants of local conservation type to be calculated.

4. Conclusions

We have described the generalisation of the concept of local frequency spectra to a class of non-linear wave equations that can be described by some local dispersion relation that depends upon the wavefield itself. This class includes such well known equations as the non-linear Schrödinger equation, the (modified) Korteweg-de Vries equation and the Burgers equation, but not an equation like the sine-Gordon equation. Furthermore the transport equations that govern these local frequency spectra have been constructed and worked out in more detail for the case in which the wave-field dependence of the dispersion relation can be separated from the dispersion relation of the linearised equation.

By taking local moments of the transport equation with respect to the frequency variable, an infinite sequence of balance equations can be derived. This infinite sequence of balance equations is in principle identical with the original wave equation. For non-linear wave equations that belong to the above-mentioned class and of which the dispersion relation of the linearised version is an entire function of the complex wavevector, the balance equations have been calculated explicitly.

It turned out that for the non-linear Schrödinger equation the successive conservation laws could be constructed straightforwardly from the balance equations in terms of Hirota derivatives. The reason that this conversion of the balance equation into conserved densities that are expressible in terms of Hirota derivatives is possible for the non-linear Schrödinger equation whereas this seems to be impossible for the other equations is unclear and should be investigated.

There are some other interesting applications of local frequency spectra with respect to non-linear wave phenomena. One aspect has already been mentioned in § 2.1, namely the study of the modulational instability of a uniform wavetrain of which the envelope is governed by a generalised non-linear Schrödinger equation. Also the long-time behaviour can thus be investigated.

Since the concept is also applicable to wave phenomena in more than one spatial dimension, it is interesting to study the transport equations, the balance equations and their possible conversion into conservation laws for multi-dimensional non-linear wave phenomena.

Finally it should be interesting to investigate the possibility of generalising the concept of local frequency spectra to discrete non-linear wave equations such as, for example, the non-linear lattice equations.

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